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Discrete and continuous exponential transforms of simple Lie groups of rank 2

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Abstract

We develop and describe continuous and discrete transforms of class functions on compact simple Lie group G as their expansions into series of uncommon special functions, called here E-functions in recognition of the fact that the functions generalize common exponential functions. The rank of G is the number of variables in the E-functions. A uniform discretization of the decomposition problem is described on lattices of any density and symmetry admissible for the Lie group G.

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1. Introduction

The aim of this paper is to generalize common exponential functions in one variable x,

$$E_m(x) := e^{imx}, \qquad m \in \mathbb{Z}, \quad x \in \mathbb{R}, \tag{1.1}$$

together with the corresponding Fourier transform,

$$f(x) = \sum_{m=0}^{\infty} b_m\, e^{imx}, \qquad b_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-imx}\, dx, \tag{1.2}$$

to any number of variables.

For close to two centuries functions (1.1) have been part of Fourier analysis. Crucial property is their pairwise orthogonality when integrated over a range $a \leqslant x \leqslant a+2\pi$ with any $a \in \mathbb{R}$. More recent but equivalent interpretation of the functions is as irreducible characters of the one-parametric unitary group $U(1)$. There is yet another interpretation of the functions (1.1), rather trivial in this case, which nevertheless is the departure point for our generalization. It is presented in the example of section 3.

The simplest possible n-dimensional generalization of exponential functions is based on the n-fold product $U(1) \times U(1) \times \cdots \times U(1)$. Corresponding functions are products of n copies of $E_m(x)$, each depending on its own continuous variable x and its own lattice variable m. It is undoubtedly useful and it is frequently used. However, we are not concerned with such a generalization in this paper.

Underlying symmetry group of E-functions in this paper is any compact semisimple Lie group G of rank n in general. The rank of G is the number of continuous variables. Here our considerations are focused on the three simple Lie groups of rank 2, namely $SU(3)$, $O(5)$ (or $Sp(4)$) and $G(2)$. Our aim is to describe the three cases in a ready-to-use form.

There are two papers dealing briefly with E-functions: their definition appeared in [1] and their orthogonality is proven in [2]. We know of no other attempt in the literature to generalize exponential functions to more than one variable.

First however let us underline a close relation of the E-functions of this paper with the C- and S-functions of [3–7]. All three families of functions are based on a semisimple compact Lie group G. They are constructed by summation of appropriate products of $U(1)$ characters over an orbit of a relevant finite group. It is the Weyl group W of G in the case of C- and S-functions, and it is the even subgroup $W_e \subset W$ of G in the case of the E-functions. Furthermore, due to the relation of the three families to the group elements of the maximal torus of the underlying Lie group, one gets the symmetries of the functions with respect to both affine Weyl group W^{aff} and even affine Weyl group W_e^{aff}.

For comparison with (1.1), let us point out that one-dimensional C- and S-functions are (up to a normalization) the familiar trigonometric functions

$$C_m(x) \sim \cos mx, \qquad S_m(x) \sim \sin mx, \qquad m \in \mathbb{Z}^{\geqslant 0}, \quad x \in \mathbb{R}.$$

Within either C- or S-family, the functions are pairwise orthogonal in $a \leqslant x \leqslant a + \pi$. The underlying Lie group is $SU(2)$ for both families.

Recently, C-functions were studied relatively extensively. Their properties are reviewed in [6] (see also references therein), similarly the S-functions are found in [1, 2, 7]. Let us as well point out that some properties of functions symmetrized over a finite group in general are described by Macdonald [8].

It may appear (falsely) that the expansions, based on compact semisimple Lie groups, impose severely constraining requirements on functions amenable to such expansions. Typically one is interested in expansions of a function given on a finite region, say R, of a real Euclidean space $\mathbb{R}^n$ either as a continuous function ('continuous data') or by its values at lattice points in R ('digital data'). In our approach, one first needs to choose a semisimple Lie group of rank n whose weight lattice has the same geometric structure as the lattice of the data (there is always at least one such group). Then the region R is inserted into the fundamental region F of the chosen Lie group $R \subset F$. In the case of digital data, a unique aspect of our method is the easy possibility of matching the density of the data points in R by the density of a grid F_M in F. More precisely, the positive integer M selects a finite Abelian subgroup of the Lie group such that its conjugacy classes are represented by the points of F_M.

The families of C-, S- and E-functions have a number of properties in common. In addition to their orthogonality, when integrated over a finite region F of an Euclidean space $\mathbb{R}^n$, they are also discretely orthogonal when summed up over a discrete grid $F_M \subset F$. Discrete orthogonality of C-functions in general is the main content of [9]. Furthermore, the lattice, obtained when F_M is extended to the whole space $\mathbb{R}^n$, is setup uniformly for all three families. Density of such a lattice is specified by a positive integer M. Functions of the three families are eigenfunctions of the same Laplace operator, namely the one appropriate for the group G, differing mainly by their behaviour at the boundary of F. Their eigenvalues are known explicitly for all n and all three families. Their products are decomposable into their sums. The functions can be built up recursively (in lattice variables), for any number of variables $x \in \mathbb{R}^n$, by a judicious choice of the lowest few and by their successive multiplication. In principle, they could be also built recursively in the points of F_M (for the same lattice variable), using the fact that the points of F_M stand for conjugacy classes of a finite Abelian group, although it could be a laborious way to do it.

Discrete orthogonality of C-functions, see [9], were exploited in challenging mathematical applications, see [10] and references therein. Immediate motivation of our current interest in the three families of functions arose as a result of the observation made in [11–13] that continuous extensions of the (finite) discrete expansions of functions on F_M smoothly interpolate digital data between discrete points. Such an observation is strongly supported by numerous convincing examples and qualitative arguments in the case of C-functions and certainly carries over to S-functions. However, a quantitative demonstration has yet to be made even in those cases.

In two dimensions practical need to interpolate digital data leads to development of a number of sophisticated interpolation methods. Comparison of such methods with ours depends on the model functions one compares it with. In general, one may say that the precision of the best interpolation methods is comparable to ours. However, unlike our approach, none of these methods readily generalizes to higher dimensions. In our case, all one needs is to replace one compact semisimple Lie group of rank 2 by another one of rank $n < \infty$.

Generalization of the E-transform from one to more dimensions in the case of a semisimple Lie group which is not simple, say $G = G_1 \times G_2$, presents two interesting options, each worth to be explored. The fundamental region F^e, where the expansion takes place, and the expansion functions are different. In spite of that one has in both cases the continuous and discrete orthogonality of the functions in F^e. The simpler option of the two is followed up in this paper.

In section 2, we briefly recall the definition of the Weyl group of simple (or semisimple) Lie group and its affine Weyl group and their basic properties. Also we define the subgroup of even elements of the Weyl group W_e and the corresponding even affine Weyl group W_e^{aff} together with their root lattice, fundamental region, etc. In section 3, we introduce E-functions $E_\lambda(x)$ of a Weyl group. The functions are specified by a given point $\lambda \in \mathbb{R}^n$. Their W_e-invariance is shown. Also, analogously to the case of both C- and S-functions, E-functions are orthogonal over the fundamental region $F^e \subset \mathbb{R}^n$ of W_e. The general method of expansion of a function on F into the sum of orthogonal E-functions is given. We illustrate it on the case of the rank-one Lie group A_1. In section 4, the E-functions together with their continuous transforms are described for the three simple Lie groups of rank 2. Discrete orthogonality of E-functions in general is the content of section 5, while in section 6 pertinent properties are described for exploitation of continuous extensions of discrete E-expansions of functions on the fundamental region F^e for the simple Lie groups of rank 2. The decomposition of the product of E-functions for these groups is the subject of section 7. Finally, in section 8 we introduce central splitting of functions given on F or F_M into the sum of s functions, where s is the order of the centre of the corresponding Lie group. Each component function has simpler E-functions expansions. Concluding remarks and some related problems are brought forward in section 9.

2. Weyl group, its even subgroup and their affinizations

Let r_i be reflection transformation of $\mathbb{R}^n$ with respect to $(n-1)$-dimensional subspace containing the origin. Consider finite groups W generated by n such reflections $r_1, r_2, \ldots, r_n$.

For any point $\lambda \in \mathbb{R}^n$ we define the orbit $W(\lambda)$ of the point λ under the action of W as the set of all different points of the form $\omega\lambda, \omega \in W$. Then the corresponding orbit function is the following:

$$C_\lambda(x) = \sum_{\mu \in W(\lambda)} \mathrm{e}^{2\pi \mathrm{i}\langle \mu, x\rangle}, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $\langle\, ,\rangle$ is a scalar product in $\mathbb{R}^n$. Note that for $n = 1$ we have $C_\lambda(x) = 2\cos(\pi m x)$, where $m \in \mathbb{Z}^{\geqslant 0}$ and $x \in \mathbb{R}$.

In this paper, we consider E-functions which are orbit functions corresponding to symmetry group W_e of even elements of Weyl groups of simple (or semisimple) Lie group. Below we recall some basic definition about both Lie groups and corresponding Weyl groups. For further information about both simple Lie groups and Weyl groups we refer to the books [15, 16].

The Weyl group W of any simple (or semisimple) Lie group is specified by its Coxeter–Dynkin diagrams. The diagram is a concise way to give a certain non-orthogonal basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ in $\mathbb{R}^n$. Each node of the diagram is associated with a basis vector α_k, called the simple root of the Lie group. Acting by elements of the Weyl group W upon simple roots, we obtain a finite system of vectors, which is invariant with respect to W. A set of all such vectors is called the root system Δ associated with a given Coxeter–Dynkin diagram. The set of all linear combinations

$$Q = \left\{ \sum_{i=1}^{n} a_i \alpha_i \,\middle|\, a_i \in \mathbb{Z} \right\} = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i \tag{2.2}$$

is called the root lattice. Relative lengths and angles between simple roots of the basis Π are specified in terms of the elements of the Cartan matrix $C = (c_{ij})_{i,j=1}^n$, where

$$c_{ij} = \frac{2\langle \alpha_i | \alpha_j\rangle}{\langle \alpha_j | \alpha_j \rangle} = \langle \alpha_i | \check{\alpha}_j \rangle, \qquad \text{for} \quad i, j = 1, \ldots, n.$$

Here $\check{\alpha}_j$ is the simple root of the dual root system $\check{\Delta} = \{\check{\alpha} = 2\alpha/\langle\alpha, \alpha\rangle | \alpha \in \Delta\}$. Denote by $\check{Q}$ the corresponding coroot lattice. Absolute length for the roots is chosen by an additional convention, namely that the longer roots of Π satisfy $\langle \alpha | \alpha \rangle = 2$. In addition to the α-basis, it is convenient to introduce the basis of fundamental weights $\omega_1, \ldots, \omega_n$. The ω-basis and α-basis are related by the inverse of the Cartan matrix

$$\omega_j = \sum_{k=1}^{n} (C^{-1})_{jk} \alpha_k.$$

Analogously to the root lattice we introduce the weight lattice

$$P = \left\{ \sum_{i=1}^{n} a_i \omega_i \,\middle|\, a_i \in \mathbb{Z} \right\} = \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i.$$

We also define the set of dominant weights P^+ and the set of strictly dominant weights P^{++}

$$P^+ = \mathbb{Z}^{\geqslant 0}\omega_1 + \mathbb{Z}^{\geqslant 0}\omega_2 + \cdots + \mathbb{Z}^{\geqslant 0}\omega_n \supset P^{++} = \mathbb{Z}^{>0}\omega_1 + \mathbb{Z}^{>0}\omega_2 + \cdots + \mathbb{Z}^{>0}\omega_n. \tag{2.3}$$

For each $\alpha \in \Delta$ and integer k we define the hyperplane

$$H_{\alpha,k} = \{t \in \mathbb{R}_n | \langle t, \alpha \rangle = k\}$$

and the associated reflection $r_{\alpha,k}$, in the hyperplane $H_{\alpha,k}$,

$$r_{\alpha,k}\, x = x - \langle \alpha, x \rangle \check{\alpha} + k\check{\alpha}. \tag{2.4}$$

The finite Weyl group W is generated by $r_{\alpha,0}$, $\alpha \in \Pi$. Since the action of W on Π gives the root system $W\Pi = \Delta$, W can be extended to the affine Weyl group W^{aff}, the group generated by $r_{\alpha,k}$ for all $\alpha \in \Pi$ and $k \in \mathbb{Z}$. W^{aff} is an infinite group such that

$$W^{\mathrm{aff}} = \check{Q} \rtimes W. \tag{2.5}$$

It is the semidirect product of its subgroups W and the invariant Abelian subgroup $\check{Q}$, the coroot lattice, for proof see [16].

For any Weyl group there exists a unique highest root

$$\xi_h = \sum_{i=1}^{n} m_i \alpha_i \equiv \sum_{i=1}^{n} q_i \check{\alpha}_i.$$

Coefficients m_i and q_i are called marks and comarks correspondingly and could be found in [17].

Finally, for any affine Weyl group we introduce its fundamental domain (or region) $F \subset \mathbb{R}^n$ as the convex hull of $\left\{0, \frac{\check{\omega}_1}{q_1}, \ldots, \frac{\check{\omega}_n}{q_n}\right\}$. By definition F is closed. If G is not simple then its fundamental region is the Cartesian product of fundamental regions of its simple components.

2.1. Subgroup of even elements of the Weyl group W_e and its affine group W_e^{aff}

Let W be a Weyl group of simple Lie group. This group is generated by reflection transformations r_i, $i = 1, \ldots, n$. We consider the subset of W

$$W_e = \langle r_{i_1} \ldots r_{i_p} \mid p \text{ is even}, i_j \in \{1, \ldots, n\} \rangle,$$

i.e., the set of elements generated by even number of reflections or the elements of W of even length. Obviously, W_e forms a finite normal subgroup of W of index 2, such that

$$W = W_e \dot{\cup} \bigcup_{i=1}^{n} r_i W_e \equiv W_e \dot{\cup} r_i W_e, \qquad |W_e| = \frac{1}{2}|W|. \tag{2.6}$$

The index i in the last equation is arbitrary, since $r_j r_i{}^{-1} \in W_e$ for any $i, j \in \{1, \ldots, n\}$. For any point λ in $\mathbb{R}^n$ we denote by $W_e(\lambda)$ its orbit with respect to the action of the even Weyl group. Every λ is contained in precisely one W_e-orbit. From [6] it follows that each original W-orbits contains a unique $\mu \in P^+$. By (2.6) we obtain that each W_e-orbit contains a unique element belonging to $P_e := P^+ \cup r_i P^{++}$. The W- and W_e-orbits are in the following correspondence with the orbits of original Weyl groups:

$$W(\lambda) = \begin{cases} W_e(\lambda) \cup W_e(r_i\lambda), & \text{if } \lambda \in P^{++} \quad \text{for some } i \in \{1, \ldots, n\}, \\ W_e(\lambda), & \text{if } \lambda \in P^+ \backslash P^{++}. \end{cases} \tag{2.7}$$

In particular, if we denote by $|W_e(\lambda)|$ the size of the orbit $W_e(\lambda)$, then $|W_e(\lambda)|$ is either equal to $|W(\lambda)|$ or to $\frac{1}{2}|W(\lambda)|$.

Consider the original affine group W^{aff} generated by $r_{\alpha,k}$ defined in (2.4). Then the even affine group is the subgroup of words of even length in $r_{\alpha,k}$ of W^{aff}, i.e.

$$W_e^{\mathrm{aff}} = \left\langle r_{\alpha_{i_1},k_1} \ldots r_{\alpha_{i_p},k_p} \middle| p \text{ is even}, \alpha_{i_j} \in \Pi, k_i \in \mathbb{Z} \right\rangle. \tag{2.8}$$

As in the case of original affine Weyl group we have the following relation between W_e^{aff} and W_e:

$$W_e^{\mathrm{aff}} = \check{Q} \rtimes W_e. \tag{2.9}$$

Indeed, the subgroup of W_e^{aff} generated by $r_{\alpha,0}$ coincides with W_e, therefore $W_e < W_e^{\mathrm{aff}}$. For any element $d \in \mathbb{R}^n$ define a translation

$$\tau(d)x = x + d, \qquad x \in \mathbb{R}^n.$$

For any two $d, d' \in \mathbb{R}^n$, $\tau(d)\,\tau(d') = \tau(d + d')$, therefore we may identify $\check{Q}$ with a group of translations on $\mathbb{R}^n$. Since

$$r_{\alpha,k} = \tau(k\check{\alpha})r_{\alpha,0} \tag{2.10}$$

we obtain that for $\alpha \in \Pi$ and $k \in \mathbb{Z}$ $\tau(k\check{\alpha}) = r_{\alpha,k}r_{\alpha}^{-1} \in W_e^{\mathrm{aff}}$. In particular for any $i = 1, \ldots, n$ $\tau(\check{\alpha}_i) \in W_e^{\mathrm{aff}}$ and therefore $\check{Q} \in W_e^{\mathrm{aff}}$. Further, from (2.10) follows that $W_e^{\mathrm{aff}} = W_e\,\check{Q}$. Also since any non-zero element of $\check{Q}$ has infinite order and any non-zero element of W_e has finite order we obtain that $W_e \cap \check{Q} = \mathrm{id}$. Finally, we have to show that subgroup $\check{Q}$ is normal in W_e^{aff}. Indeed, for any $w \in W_e \subset W$ and any $d \in \check{Q}$ we have $w\,\tau(d)w^{-1} = \tau(wd)$.

Using (2.6) we also can define now the fundamental domain of W_e^{aff} being a set

$$F^e = F \cup r_i F. \tag{2.11}$$

3. Definition of E-functions, their relations to C-functions

We start with the definition (2.1) of C-functions. The C-function $C_\lambda(x)$ is the contribution to an irreducible character from the orbit $W(\lambda)$, $\lambda \in P^+$. If in (2.1) we restrict ourselves to the orbit $W_e(\lambda)$ instead of the orbit of W, we obtain the E-function $E_\lambda(x)$:

$$E_\lambda(x) = \sum_{\mu \in W_e(\lambda)} \mathrm{e}^{2\pi\mathrm{i}\langle\mu,x\rangle}, \qquad \text{for} \quad x \in \mathbb{R}^n, \quad \lambda \in P. \tag{3.1}$$

The C-functions appeared in [9, 6] under the name 'orbit functions'. Their many properties, very useful for applications, were extensively studied in [1–6]. In this section, we formulate analogous properties of E-functions.

To start, both families of C- and E-functions are based on semisimple Lie algebra, the rank of the algebra is the number of variables. They are given as the finite sums of exponential functions, therefore they are continuous and have derivatives of all orders in $\mathbb{R}^n$.

3.1. W_e- and W_e^{aff}-invariance of E-functions

The C-functions (2.1) are invariant under the action of W. For any $\lambda \in P$, $E_\lambda(x)$ is invariant under the action of W_e. Indeed,

$$E_\lambda(wx) = \sum_{\mu \in W_e(\lambda)} \mathrm{e}^{2\pi\mathrm{i}\langle\mu,wx\rangle} = \sum_{\mu \in W_e(\lambda)} \mathrm{e}^{2\pi\mathrm{i}\langle w^{-1}\mu,x\rangle} = E_\lambda(x) \qquad \text{for any } w \in W_e, \tag{3.2}$$

since the scalar product $\langle\,,\rangle$ is invariant with respect to W and $W_e(\mu) = W_e(w^{-1}\mu)$.

The C-function corresponding to $\lambda \in P$ is invariant under the action of W^{aff}. Let us show that E-functions for $\lambda \in P$ are invariant with respect to W_e^{aff}. Since $W_e^{\mathrm{aff}} = \check{Q} \rtimes W_e$ it is enough to show invariance of $E_\lambda(x)$ with respect to any translation $\tau(d)$, $d \in \check{Q}$. For $\lambda \in P$ any $\mu \in W_e(\lambda)$ also belongs to P hence

$$\mathrm{e}^{\langle\mu,\tau(d)x\rangle} = \mathrm{e}^{\langle\mu,d\rangle+\langle\mu,x\rangle} = \mathrm{e}^{\mathrm{integer}+\langle\mu,x\rangle} = \mathrm{e}^{\langle\mu,x\rangle}. \tag{3.3}$$

Therefore we may consider E-functions only on the fundamental domain F^e. The values on other points of $\mathbb{R}^n$ are determined by using the action of W_e^{aff} on F^e.

3.2. Relation between E- and C-functions

The original Weyl group also acts on the E-functions. By (2.7) for any $\lambda \in P^{+}\backslash P^{++}$ we obtain that $E_{\lambda}(r_i x) = E_{\lambda}(x)$, if $\lambda \in P^{++}$ then $E_{\lambda}(r_i x) = E_{r_i\lambda}(x)$. Bringing it all together, we obtain

$$C_{\lambda}(x) = \begin{cases} E_{\lambda}(x) + E_{r_i\lambda}(x) & \text{if } \lambda \in P^{++}, \\ E_{\lambda}(x), & \text{otherwise.} \end{cases} \tag{3.4}$$

We call $\lambda \in P_e$ an intrinsic point if $\lambda \in P^{++} \cup r_i P^{++}$.

3.3. Eigenfunctions of the Laplace operator

It was shown in [6, 7] that both S- and C-functions are eigenfunctions of the same differential operator

$$L = (\alpha_1\partial_1 + \alpha_2\partial_2 + \cdots + \alpha_n\partial_n)^2 .$$

Since the matrix of scalar products of simple roots is positive defined, by a suitable choice of basis, the operator can be brought to the sum of second derivatives with positive coefficients, therefore one may call L the Laplace operator.

Here we will show that the E-functions are eigenfunctions of L as well. We parametrize elements of F^e by the coordinates in the ω-basis $x = \theta_1\omega_1 + \cdots + \theta_n\omega_n$ and denote by ∂_j the partial derivative with respect to θ_j. Consider the application of L to E-functions, we see that they also are its eigenfunctions:

$$LE_{\lambda} = -4\pi^2\langle\lambda \mid \lambda\rangle E_{\lambda}.$$

In fact, every exponential term in the functions is individually an eigenfunction of L. Since weights of one orbit are equidistant from the origin, eigenvalues of all terms in each function coincide. The explicit form of the Laplace operators L corresponding to the simple Lie group of rank 2 is given in [3, 4].

3.4. Orthogonality and E-function transforms

Both family of C- and E-functions determine a symmetrized Fourier series expansions. The proof in general for both families and both continuous and discrete cases is given in [2]. It is based on the orthogonality of E-functions (C-functions) determined by points $\lambda \in P_e$ (correspondingly $\lambda \in P^{+}$). For any $\lambda, \lambda' \in P_e$ corresponding E-functions are orthogonal on F^e with respect to Euclidean measure:

$$\int_{F^e} E_{\lambda}(x)\overline{E}_{\lambda'}(x)\,\mathrm{d}x = |F^e||W_e(\lambda)|\delta_{\lambda\lambda'}, \tag{3.5}$$

where bar means a complex conjugation and $|F^e|$ is a volume of the fundamental domain F^e. This relation follows from the orthogonality of the exponential functions for different weights λ and from the fact that each point $\mu \in P_e$ belongs to precisely one W_e-orbit. Therefore, the E-functions corresponding to the points of $\lambda \in P_e$ form an orthogonal basis in the Hilbert space of squared integrable function on F^e. Therefore, we may expand functions on F^e as sums of E-functions. Let f be a function defined on F^e then it may be written as

$$f(x) = \sum_{\lambda \in P^e} c_{\lambda} E_{\lambda}(x), \tag{3.6}$$

where c_{λ} is determined by

$$c_{\lambda} = |W_e(\lambda)||F^e|^{-1} \int_{F^e} f(x)\overline{E}_{\lambda}(x)\,\mathrm{d}x. \tag{3.7}$$

For details of the proof see [2].

3.5. Example: E-functions for A_1

The E-functions of the rank 1 simple Lie group A_1 happen to be the common exponential functions. Indeed, the Weyl group of A_1 has two elements $W = \{\mathrm{id}, r\}$, where r is the reflection in the origin of $\mathbb{R}$. The root lattice consists of all even integer points of $\mathbb{R}$, the weight lattice P is formed by all integers. The even subgroup $W_e \subset W$ is the identity element of W. Thus for any point λ, its W_e-orbit consists of a single point. Consequently, the E-function is a single exponential function:

$$E_\lambda(x) = E_m(x) \stackrel{\mathrm{def}}{=} \sum_{\mu\in W_e(\lambda)} \mathrm{e}^{2\pi\mathrm{i}\langle\mu|x\rangle} = \mathrm{e}^{\mathrm{i}\pi m x}. \tag{3.8}$$

The fundamental region $F^e(A_1) \stackrel{\mathrm{def}}{=} F(A_1) \cup rF(A_1) = [-1, 1]$ is in ω-basis. For this simple case one can directly verify the decomposition of the products:

$$E_m(x)\overline{E}_{m'}(x) = E_{m-m'}(x), \qquad \text{for } m, m' \in \mathbb{Z}. \tag{3.9}$$

Consequently we obtain that any two functions $E_m(x)$, $E_{m'}(x)$ with $m \neq m'$ are orthogonal, i.e.,

$$\int_{-1}^{1} E_m(x)\overline{E}_{m'}(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } m \neq m', \\ 2, & \text{if } m = m'. \end{cases} \tag{3.10}$$

The continuous E-transform is the expansion (3.6) of functions over $-1 \leqslant x \leqslant 1$:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m\,\mathrm{e}^{\mathrm{i}\pi m x}, \qquad c_m = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{e}^{-\mathrm{i}\pi m x}\,\mathrm{d}x. \tag{3.11}$$

4. Continuous E-transform for simple Lie groups of rank 2

Next three sections deal with the main topic of the paper, namely expansions of functions on F^e into series of E-functions and their inversion (direct and inverse E-transforms). In this section, the functions to expand, as well as the E-functions, are continuous ones. Discrete transforms are the subject of sections 5 and 6.

General structure of the continuous direct and inverse two-dimensional E-transform is the following:

$$\begin{aligned} f(x,y) &= \sum_{(a,b)\in P_e} c_{a,b} E_{(a,b)}(x,y), \\ c_{a,b} &= \frac{1}{\int_{F^e} E_{(a,b)}(x,y)\overline{E}_{(a,b)}(x,y)\,\mathrm{d}x\,\mathrm{d}y} \int_{F^e} f(x,y)\overline{E}_{(a,b)}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.1}$$

Here, the overbar indicates complex conjugation. There are four pieces of information one needs before the transform (4.1) can be applied to a given function $f(x, y)$. This information depends on the particular Lie group G. We needs to provide

(i) the infinite set P_e of points of the weight lattice P;
(ii) the finite domain $F^e \in \mathbb{R}^2$;
(iii) the functions $E_{(a,b)}(x, y)$ for $(a, b) \in P_e$;
(iv) The normalization coefficients $\int_{F^e} E_{(a,b)}(x, y)\overline{E}_{(a,b)}(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

There are three compact simple Lie groups of rank 2, namely $SU(3)$, $Sp(4) \equiv O(5)$ and $G(2)$. Also there is one semisimple compact Lie group $SU(2) \times SU(2)$ which is not simple. We are using the following notation often used to denote the corresponding Lie algebras:

$$A_2 \leftrightarrow SU(3), \qquad C_2 \leftrightarrow O(5), \qquad G_2 \leftrightarrow G(2), \qquad A_1 \times A_1 \leftrightarrow SU(2) \times SU(2).$$

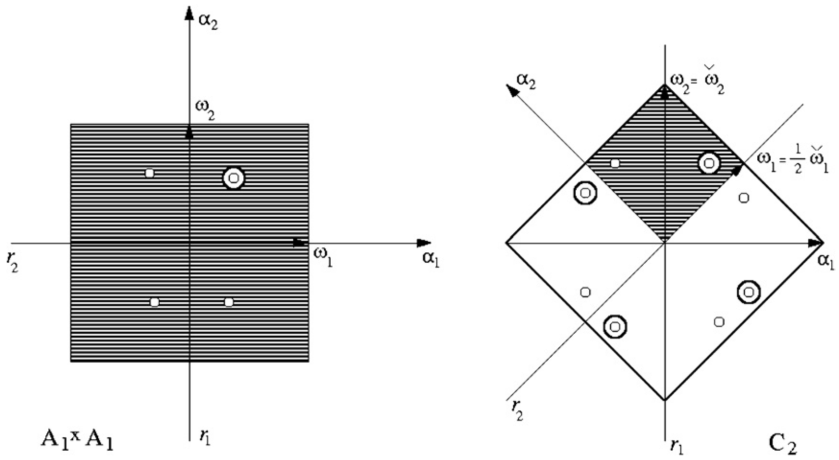

Figure 1. The simple roots, the fundamental weights, along with their dual, r_1, r_2 generators of W and the even fundamental region (shaded area) for $A_1 \times A_1$ and C_2. The dots ○ denote the points of W-orbit and dots ◯ denote the points of W_e-orbit.

4.1. The E-transforms of $A_1 \times A_1$

As we already mentioned in the introduction there are two ways to define the even Weyl group for Lie group $G = G_1 \times G_2$ which is a product of two simple groups. More on this subject are in concluding remarks, section 9. Here we give an example when $G = A_1 \times A_1$ and we use $W_e(A_1) \times W_e(A_1)$ for $W_e(A_1 \times A_1)$. Then this case becomes a simple concatenation of two cases of A_1 described in section 3.5.

Relative length and angles of the simple roots are given by the scalar products

$$\langle \alpha_1 \mid \alpha_2 \rangle = 0, \qquad \langle \alpha_1 \mid \alpha_1 \rangle = \langle \alpha_2 \mid \alpha_2 \rangle = 2.$$

Consequently, $\alpha_1 = 2\omega_1$ and $\alpha_2 = 2\omega_2$. Their dual $\check{\alpha}_k$ and $\check{\omega}_j$ coincide with α_k and ω_j. The root system $\Delta = \{\pm\alpha_1, \pm\alpha_2\}$ geometrically represents the vertices of a square of a side length 2. See figure 1 for the details.

Suppose $\lambda = a\omega_1 + b\omega_2$, where $a, b \in \mathbb{Z}$. Since $W_e(A_1)$ consists trivially of its identity element, all $W_e(A_1)$-orbits have just one weight, so that

$$E_{(a,b)}(x, y) = \mathrm{e}^{\pi \mathrm{i}(ax+by)}, \qquad a, b \in \mathbb{Z}, \quad x, y \in \mathbb{R}.$$

The fundamental region of $W_e(A_1 \times A_1)$ is a direct product of the fundamental regions of $W_e(A_1)$, i.e.,

$$F^e(A_1 \times A_1) = \{x\omega_1 + y\omega_2 \mid \text{where } -1 \leqslant x, y \leqslant 1\}. \tag{4.2}$$

Thus, we have the familiar extension of the one-dimensional transform (1.2) to two dimensions.

4.2. The E-transforms of C_2

Relative length and angles of the simple roots of C_2 are given by

$$\langle \alpha_1 \mid \alpha_2 \rangle = -1, \qquad \langle \alpha_1 \mid \alpha_1 \rangle = 1, \qquad \langle \alpha_2 \mid \alpha_2 \rangle = 2.$$

Consequently,

$$\alpha_1 = 2\omega_1 - \omega_2, \qquad \omega_1 = \alpha_1 + \tfrac{1}{2}\alpha_2, \qquad \check{\alpha}_1 = 2\alpha_1,$$

$$\alpha_2 = -2\omega_2 + 2\omega_2, \qquad \omega_2 = \alpha_1 + \alpha_2, \qquad \check{\alpha}_2 = \alpha_2.$$

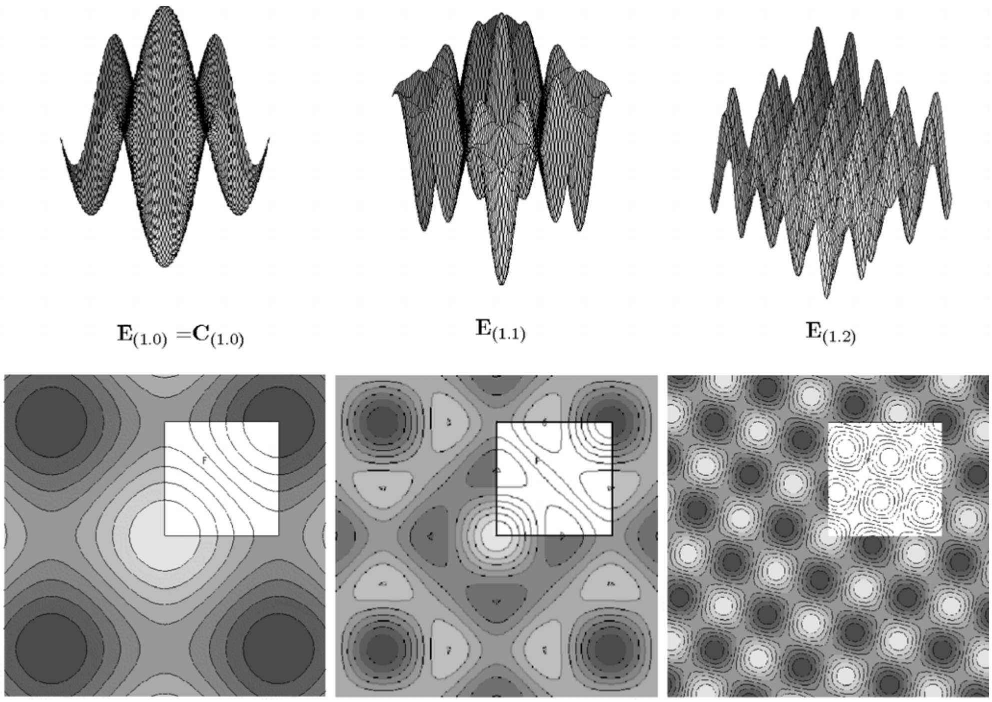

Figure 2. Examples of E-functions for C_2.

The root system $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$ geometrically represents the vertices and midpoints of a square.

The fundamental region $F^e(C_2)$ is defined as $F(C_2) \cup r_1 F(C_2)$, i.e.,

$$F^e(C_2) = \{x\check{\omega}_1 + y\check{\omega}_2 \mid \text{where } 0 \leqslant y \leqslant 1 \text{ and } 0 \leqslant 2x+y \leqslant 1\}. \tag{4.3}$$

Geometrically it is a square with vertices 0, $\frac{\check{\omega}_1}{2}$, $\check{\omega}_2$ and $\check{\omega}_2 - \frac{\check{\omega}_1}{2}$. See figure 1 for the details.

We define $P_e = P^+ \cup r_1 P^{++}$. For $\lambda = (a,b) = a\omega_1 + b\omega_2 \in P_e$, the even Weyl group orbit $W_e(\lambda) \equiv W_e(a,b)$ contains either one or four points:

$$W_e(a,b) = \begin{cases} \{(0,0)\} & \text{if } a = b = 0, \\ \{\pm(a,b), \pm(a+2b, -a-b)\} & \text{if } a^2+b^2 > 0. \end{cases}$$

According to (3.1) the E-functions of the Lie group C_2, with $\lambda = a\omega_1 + b\omega_2$ and $z = x\check{\omega}_1 + y\check{\omega}_2$, are the following:

$$E_{(0,0)}(x,y) = 1,$$
$$E_{(a,b)}(x,y) = 2\cos(\pi((2a+2b)x + (a+2b)y)) + 2\cos(\pi(2bx - ay)), \quad \text{if } a^2+b^2 > 0. \tag{4.4}$$

In particular, $E_{(a,0)}(x,y) = C_{(a,0)}(x,y)$ and $E_{(0,b)}(x,y) = C_{(0,b)}(x,y)$. Examples of E-functions for C_2 are given in figure 2.

To be uniform in both formulae, we introduce a different normalization, namely,

$$\Xi_{(a,b)}(x,y) \stackrel{\text{def}}{=} \frac{|W_e|}{|W_e(a,b)|} E_{(a,b)}(x,y), \tag{4.5}$$

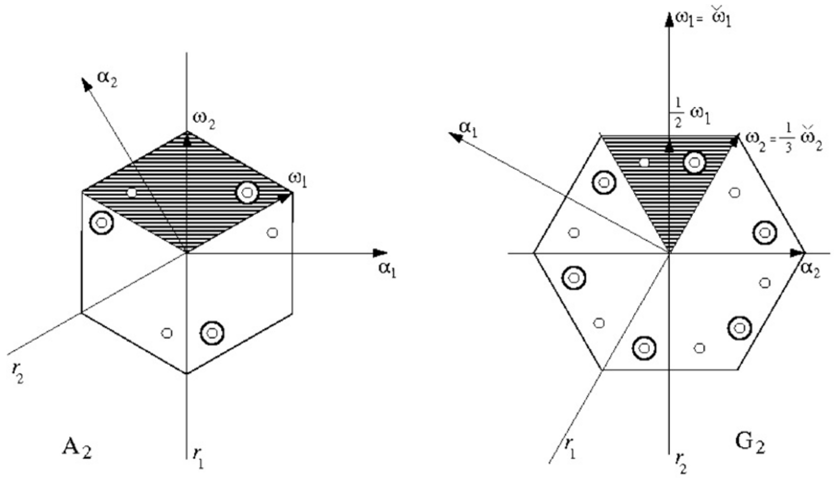

Figure 3. The simple roots, the fundamental weights, along with their dual, r_1, r_2 generators of W and the even fundamental region (shaded area) for A_2 and G_2. The dots $\circ$ denote the points of W-orbit and $\bigcirc$ the points of W_e-orbit.

where $|W_e(a,b)|$ is the number of points in $W_e(a,b)$. In C_2 case we rewrite (4.4) as

$$\Xi_{(a,b)}(x,y) = 2\cos(\pi((2a+2b)x+(a+2b)y)) + 2\cos(\pi(2bx-ay)), \qquad \text{for all } (a,b) \in P_e.$$

For any $(a,b), (c,d) \in P_e$, orthogonality property is verified directly,

$$\begin{aligned} \int_{F^e} \Xi_{(a,b)}(x,y)\overline{\Xi}_{(c,d)}(x,y)\,\mathrm{d}F^e &= \int_0^1 \mathrm{d}y \int_{-\frac{y}{2}}^{1-\frac{y}{2}} \Xi_{(a,b)}(x,y)\overline{\Xi}_{(c,d)}(x,y)\,\mathrm{d}y \\ &= \begin{cases} 0, & \text{if } a \neq c \text{ and } b \neq d, \\ 2, & \text{if } a = c \text{ or } b = d. \end{cases} \end{aligned} \tag{4.6}$$

In particular, we have $\int_{F^e} \Xi_{(a,b)}(x,y)\,\mathrm{d}F^e = 0$ for any $(a,b) \neq (0,0)$.

4.3. The E-transforms of A_2

Relative length and angles of the simple roots of A_2 are given by

$$\langle \alpha_1 \mid \alpha_2 \rangle = -1, \qquad \langle \alpha_1 \mid \alpha_1 \rangle = \langle \alpha_2 \mid \alpha_2 \rangle = 2.$$

Consequently,

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2, & \omega_1 &= \tfrac{1}{3}(2\alpha_1 + \alpha_2), \\ \alpha_2 &= -\omega_1 + 2\omega_2, & \omega_2 &= \tfrac{1}{3}(\alpha_1 + 2\alpha_2). \end{aligned}$$

The root system $\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$ geometrically represents vertices of a regular hexagon. For details see figure 3. Take any $\lambda = a\omega_1 + b\omega_2 \in P_e = P^+ \cup r_1 P^{++}$. Then the even Weyl group orbit $W_e(a,b)$ contains one or three points, namely,

$$W_e(a,b) = \begin{cases} \{(0,0)\} & \text{if } a = b = 0, \\ \{(a,b), (b,-a-b), (-a-b,a)\} & \text{if } a^2+b^2 \neq 0. \end{cases}$$

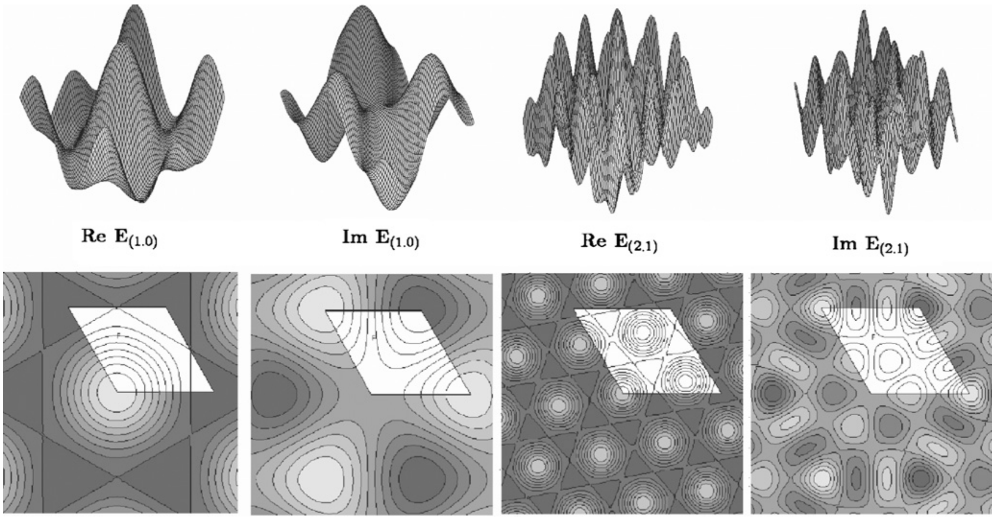

Figure 4. Real and imaginary parts of E-functions for A_2.

In particular, $\Delta = W_{(1,1)} \cup W_{(-1,-1)}$. Therefore, the E-functions of A_2, with $\lambda = a\omega_1 + b\omega_2$ and $z = x\check{\omega}_1 + y\check{\omega}_2$, are the following:

$$E_{(0,0)}(x, y) = 1,$$

$$\begin{aligned} E_{(a,b)}(x, y) = \exp\left(\frac{2\pi \mathrm{i}}{3}((2a+b)x + (a+2b)y)\right) \\ + \exp\left(-\frac{2\pi \mathrm{i}}{3}((x+2y)a + (y-x)b)\right) + \exp\left(-\frac{2\pi \mathrm{i}}{3}((x-y)a + (2x+y)b)\right). \end{aligned}$$

In particular, $E_{(a,0)}(x, y) = C_{(a,0)}(x, y)$ and $E_{(0,b)}(x, y) = C_{(0,b)}(x, y)$. See figure 4 for examples of E-functions for A_2. Using the normalization (4.5) we obtain uniform formula

$$\begin{aligned} \Xi_{(a,b)}(x, y) = \exp\left(\frac{2\pi \mathrm{i}}{3}((2a+b)x + (a+2b)y)\right) + \exp\left(-\frac{2\pi \mathrm{i}}{3}((x+2y)a + (y-x)b)\right) \\ + \exp\left(-\frac{2\pi \mathrm{i}}{3}((x-y)a + (2x+y)b)\right). \end{aligned} \tag{4.7}$$

The fundamental region $F^e(A_2)$ is a union of original fundamental region for Weyl group with its reflection with respect to r_1, i.e.,

$$F^e(A_2) = \{x\omega_1 + y\omega_2 \mid \text{where } 0 \leqslant y \leqslant 1 \text{ and } 0 \leqslant x + y \leqslant 1\}. \tag{4.8}$$

Geometrically it is a rhombus with vertices 0, ω_1, ω_2 and $\omega_2 - \omega_1$ (see figure 3).

For any $(a, b), (c, d) \in P_e$, orthogonality property of E-functions of A_2 can be verified directly,

$$\begin{aligned} \int_{F^e} \Xi_{(a,b)}(x, y)\overline{\Xi}_{(c,d)}(x, y)\,\mathrm{d}F^e = \frac{1}{\sqrt{3}}\int_0^1 \mathrm{d}y \int_{-y}^{1-y} \Xi_{(a,b)}(x, y)\overline{\Xi}_{(c,d)}(x, y)\,\mathrm{d}y \\ = \begin{cases} 0, & \text{if } a \neq c \text{ or } b \neq d, \\ \sqrt{3}, & \text{if } a = c \text{ and } b = d. \end{cases} \end{aligned} \tag{4.9}$$

4.4. The E-transform of G_2

Relative length and angles of the simple roots of G_2 are given by

$$\langle\alpha_1 \mid \alpha_2\rangle = -1, \qquad \langle\alpha_1 \mid \alpha_1\rangle = 2, \qquad \langle\alpha_2 \mid \alpha_2\rangle = \tfrac{2}{3}.$$

Then, the relation between simple roots and weights is

$$\begin{aligned} \alpha_1 &= 2\omega_1 - 3\omega_2, & \omega_1 &= 2\alpha_1 + 3\alpha_2, & \check{\omega}_1 &= \omega_1, \\ \alpha_2 &= -\omega_1 + 2\omega_2, & \omega_2 &= \alpha_1 + 2\alpha_2, & \check{\omega}_2 &= 3\omega_2. \end{aligned}$$

There are 12 roots in $\Delta(G_2)$, namely the following:

$$\Delta = \{\pm(2\alpha_1 + 3\alpha_2), \pm(\alpha_1 + 3\alpha_2), \pm(\alpha_1 + 2\alpha_2), \pm(\alpha_1 + \alpha_2), \pm\alpha_1, \pm\alpha_2\},$$

geometrically the roots are vertices of a regular hexagonal star (see figure 3).

The fundamental region $F^e(G_2)$ is $F(G_2) \cup r_2 F(G_2)$:

$$F^e(G_2) = \{x\check{\omega}_1 + y\check{\omega}_2 \mid \text{where } 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant 2x + 3y \leqslant 1\}.$$

It is a triangle with vertices 0, $\frac{\check{\omega}_2}{3}$ and $\frac{\check{\omega}_1}{2} - \frac{\check{\omega}_2}{3}$ (see figure 3). Note that for G_2 we choose the reflection with respect to r_2. Therefore we also have to redefine $P^e(G_2) = P^+ \cup r_2 P^{++}$.

Let $\lambda = a\omega_1 + b\omega_2 \in P_e$. Then the even Weyl group orbit $W_e(\lambda) \equiv W_e(a, b)$ contains one or six points. More precisely,

$$W_e(a, b) = \begin{cases} \{(0, 0)\} & \text{if } a = b = 0, \\ \{\pm(a, b), \pm(2a + b, -3a - b), \pm(-a - b, 3a + 2b)\} & \text{if } a^2 + b^2 \neq 0. \end{cases}$$

The E-functions (4.5) of G_2, with $\lambda = a\omega_1 + b\omega_2$ and $z = x\check{\omega}_1 + y\check{\omega}_2$, are the following:

$$\Xi_{(a,b)}(x, y) = 2\cos(2\pi((2a + b)x + (3a + 2b)y)) + 2\cos(2\pi(ax + (3a + b)y)) \tag{4.10}$$

$$+ 2\cos(2\pi((a + b)x + by)), \qquad (a, b) \in P_e. \tag{4.11}$$

See figure 5 for examples of E-functions for G_2.

Orthogonality of E-functions of G_2 can be verified, for any $(a, b), (c, d) \in P_e$,

$$\begin{aligned} \int_{F^e} \Xi_{(a,b)}(x, y)\overline{\Xi}_{(c,d)}(x, y)\,\mathrm{d}F^e &= \sqrt{3}\int_0^1 \mathrm{d}x \int_{-\frac{2x}{3}}^{\frac{1-2x}{3}} \Xi_{(a,b)}(x, y)\overline{\Xi}_{(c,d)}(x, y)\,\mathrm{d}y \\ &= \begin{cases} 0, & \text{if } a \neq c \text{ or } b \neq d, \\ 2\sqrt{3}, & \text{if } a = c \text{ and } b = d. \end{cases} \end{aligned} \tag{4.12}$$

5. A discrete *E*-function transforms

We introduce the essentials of the discrete finite E-function transform. This transform can be used, for example, to interpolate values of a function $f(x)$ between its given values on a lattice $F^e_M \subset F$. The discretization of E-functions closely parallels that of both C- and S-functions [3–5]. All the details of the proof for the discrete finite E-transforms are found in [2]. Recall (2.2) that the lattice $\check{Q}$ is a discrete W-invariant subset of $\mathbb{R}^n$. Then for any positive integer M the set

$$T_M = \frac{1}{M}\check{Q}/\check{Q} = \bigcup_{i=1}^{n} \frac{d_i\check{\alpha}_i}{M}, \qquad d_i = 0, \ldots, M - 1 \tag{5.1}$$

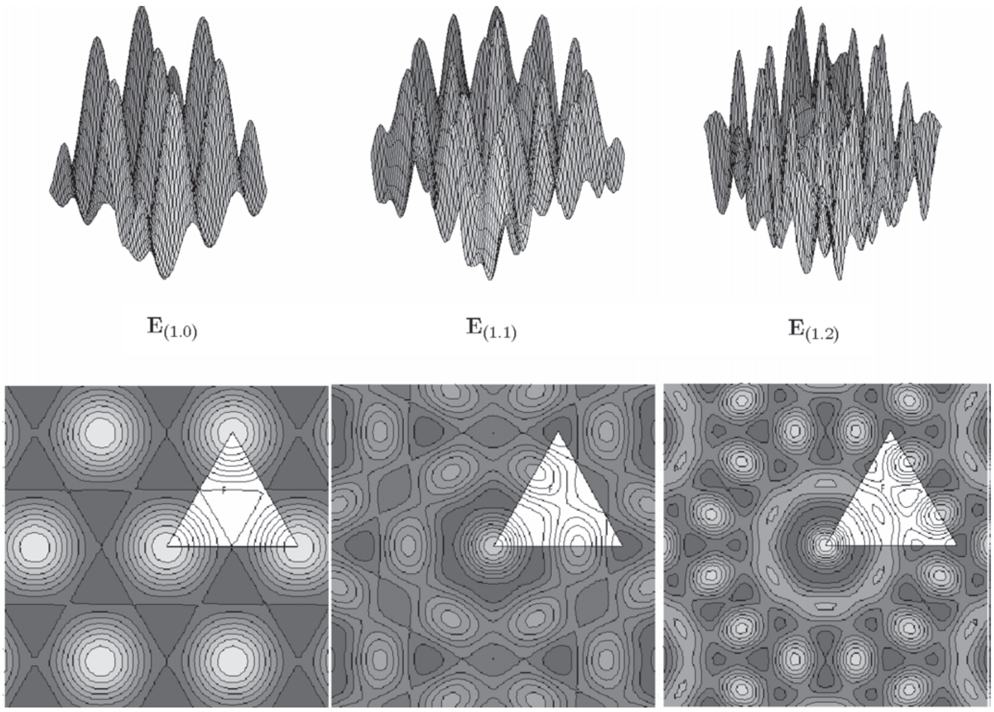

Figure 5. The E-functions for G_2.

is finite and W-invariant. Moreover, T_M forms the Abelian subgroup of the maximal torus, generated by the elements of order M, of simple compact Lie group corresponding to W. One has the basic discrete orthogonal relation on T_M for $\lambda, \mu \in P$:

$$\sum_{s\in T_M} \mathrm{e}^{2\pi\langle\lambda,s\rangle}\,\overline{\mathrm{e}^{2\pi\langle\mu,s\rangle}} = \begin{cases} |T_M|, & \text{if } \lambda_{|T_M} = \mu_{|T_M}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

We define the equidistant grid F_M^e of points in the fundamental region F^e, namely

$$\begin{aligned} F_M^e &\stackrel{\text{def}}{=} F_M \cup r_i(F_M) \\ &= \left\{ s = \frac{s_1}{M}\check{\omega}_1 + \cdots + \frac{s_n}{M}\check{\omega}_n, \, |s_i \in \mathbb{Z}^{\geqslant 0}, \sum_{i=1}^{n} s_i m_i \leqslant M \right\} \cup \{ r_i(s) \mid s \in F_M \} \end{aligned} \tag{5.3}$$

where m_i's are comarks, i.e. the coefficients of the highest root ξ_h.

For any two functions f, g given by their values on some F_M^e we introduce a bilinear form

$$\langle f|g\rangle_M \stackrel{\text{def}}{=} \sum_{s\in F_M^e} \varepsilon_s f(s)\overline{g(s)}. \tag{5.4}$$

The coefficients ε_s in the sum over F_M^e are equal to the number of points in the torus T_M that are conjugate to the point $s \in F_M^e$. By (5.2) for any M positive integer there exists a finite set Λ_M of P_e such that for any two $\lambda, \lambda' \in \Lambda_M$

$$\langle \Xi_\lambda | \Xi_{\lambda'} \rangle_M = \sum_{s\in F_M^e} \varepsilon_s \Xi_\lambda(s)\overline{\Xi(s)} = \delta_{\lambda,\lambda'} \langle \Xi_\lambda | \Xi_\lambda \rangle_M. \tag{5.5}$$

As a consequence of the orthogonality property (5.5), we get the following decomposition for any function $f(s)$ with known values on points of F_M^e. Indeed, if $f(s)$ is given

$$f(s) = \sum_{\Xi_\lambda \in \Lambda_M} d_\lambda \Xi_\lambda(s). \tag{5.6}$$

Then using the orthogonality property (5.5) we may calculate d_λ as

$$d_\lambda = \frac{\langle f | \Xi_\lambda \rangle_M}{\langle \Xi_\lambda | \Xi_\lambda \rangle_M}. \tag{5.7}$$

Once d_λ of the original decomposition (5.6) were calculated, one can extend discrete variables in F_M to continues ones:

$$f_{\mathrm{cont}}(x) \stackrel{\mathrm{def}}{=} \sum_{\Xi_\lambda \in \Lambda_M} d_\lambda \Xi_\lambda(x). \tag{5.8}$$

It turns out that the function $f_{\mathrm{cont}}(x)$ smoothly interpolates the values of $f(s)$, while coinciding with it at the points of F_M^e.

Note that to find coefficients ε_s one may use the corresponding C-function coefficients $c_s, s \in F_M$, which is equal to the number of point in T_M that are congruent to s. Indeed, by (2.11) $F^e = F \cup r_i F$ for some $i \in \{1, \ldots, n\}$. Then for $s = (s_1, \ldots, s_n) \in F_M$

$$\varepsilon_s = \begin{cases} \frac{1}{2} c_s, & \text{if } s_i \neq 0, \\ c_s, & \text{if } s_i = 0. \end{cases} \tag{5.9}$$

5.1. *Example: discretization of A_1*

Here we give the description of the A_1 version of the discrete orthogonality of the E-functions.

First, we fix $M \in \mathbb{N}$, which determines an equidistant grid of $2M + 1$ points F_M:

$$F_M^e = \left\{ -1, -\frac{M-1}{M}, \ldots, -\frac{1}{M}, 0, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M-1}{M}, 1 \right\}. \tag{5.10}$$

The scalar product in the space of functions defined on F_M^e is

$$\langle f \mid h \rangle_M \stackrel{\mathrm{def}}{=} \sum_{s \in T_M} f(s) h(s) = \sum_{s \in F_M^e} c_s f(s) h(s). \tag{5.11}$$

For any $s \neq \pm 1$ there is no other point of T_M which is conjugated to s therefore $\varepsilon_s = 1$. The point $s = 1$ is conjugate to $s = -1$ and only one of them belongs to T_M thus $\varepsilon_{-1} = \varepsilon_1 = \frac{1}{2}$. Analogously to the continuous case we obtain the discrete orthogonality property of the E-functions over F_M^e:

$$\langle E_m \mid E_{m'} \rangle_M = \begin{cases} 2M, & \text{if } m = m' \bmod 2M, \\ 0, & \text{if } m \neq m' \bmod 2M. \end{cases} \tag{5.12}$$

Let $f(s)$ be a function with known real values on F_M and be decomposed as follows:

$$f(s) = \sum_{k=-M}^{M} d_k E_k(s), \qquad s \in F_M^e. \tag{5.13}$$

Then we can compute the coefficients d_k from

$$\langle f \mid E_k \rangle_M = \sum_{s \in F_M^e} \varepsilon_s f(s) E_k(s) = \begin{cases} 4M d_k, & \text{if } k = -M \text{ or } k = M, \\ 2M d_k, & \text{if } k = -M+1, \ldots, M-1. \end{cases} \tag{5.14}$$

After the coefficients d_k have been calculated, one can replace s in (5.13) by the continuous variable x:

$$f_{\text{cont}}(x) \stackrel{\text{def}}{=} \sum_{k=-M}^{M} d_k E_k(x), \qquad \text{where} \qquad x \in \mathbb{R}. \tag{5.15}$$

At $x = s \in F_M^e$, the continuous function $f_{\text{cont}}(x)$ coincides with $f(s)$.

6. Discretization of two-dimensional transforms

This section contains all the details of the exploration of the method of finite E-function transform corresponding to the simple Lie groups of rank 2. General structure of the discrete two-dimensional E-transform is the following: for any function f given on the discrete grid F_M^e

$$f_{\text{cont}}(x, y) = \sum_{(a,b)\in\Lambda_M} d_{(a,b)} \Xi_{(a,b)}(x, y), \qquad d_{(a,b)} = \frac{\langle f | \Xi_{(a,b)} \rangle_M}{\langle \Xi_{(a,b)} | \Xi_{(a,b)} \rangle_M}. \tag{6.1}$$

Here $\langle\, ,\rangle_M$ denotes the Hermitian form (5.4). For the particular Lie group G besides the corresponding E-functions there are four other data one needs to perform transform (6.1):

(i) the finite grid $F_M^e \subset F^e$;
(ii) the coefficients ε_s for $s \in F_M^e$;
(iii) the finite subset Λ_M of P_e;
(iv) the normalization coefficients $\langle \Xi_{(a,b)} | \Xi_{(a,b)} \rangle_M$, $(a, b) \in \Lambda_M$.

6.1. Discretization in the case of C_2

First we describe the grid F_M^e as in (5.10). Since the highest root of C_2 is $2\alpha_1 + \alpha_2$

$$F_M^e \stackrel{\text{def}}{=} \left\{ \left(\frac{s_1}{M}, \frac{s_2}{M} \right) \middle| s_0, s_1, s_2 \in \mathbb{Z}^{\geqslant 0}, s_0 + 2s_1 + s_2 = M > 0 \right\}$$
$$\cup \left\{ \left(\frac{-s_1}{M}, \frac{s_2 + 2s_1}{M} \right) \middle| s_2 \neq 0 \right\}.$$

See figure 6 for F_M^e, $M = 3, 4$.

The coefficients c_s for C_2 are found in [3]. By (5.9)

$$\varepsilon_s \equiv \varepsilon_{(\frac{s_1}{M}, \frac{s_2}{M})} = \begin{cases} 1, & \text{if } s_1 = 0 \text{ and } s_2 = 0, M, \\ & \text{or } s_2 = 0 \text{ and } s_1 = \frac{M}{2}, \\ & \text{or } s_1 = -\frac{M}{2} \text{ and } s_2 = M, \\ 2, & \text{if } 2s_1 + s_2 = 0 \text{ and } 0 < s_2 < M, \\ & \text{or } 2s_1 + s_2 = M \text{ and } 0 < s_2 < M, \\ & \text{or } s_2 = 0 \text{ and } 0 < s_1 < \frac{M}{2}, \\ & \text{or } s_2 = M \text{ and } -\frac{M}{2} < s_1 < 0, \\ 4, & \text{if } 0 < s_2 < M \text{ and } 0 < 2s_1 + s_2 < M. \end{cases} \tag{6.2}$$

The finite set $\Lambda_M = \{(a, b) \in P_e \,|0 < a + 2b \leqslant M, 0 \leqslant a < M\}$. Then for any $(a, b) \neq (a', b') \in \Lambda_M$,

$$\langle \Xi_{(a,b)} \mid \Xi_{(a',b')} \rangle_M = 0,$$

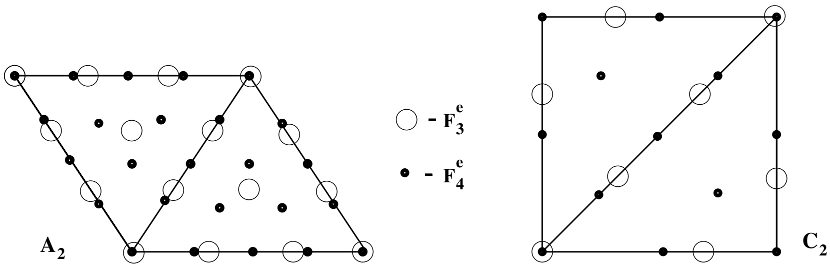

Figure 6. The lattice points of F_3^e, F_4^e in the fundamental region F^e for C_2 and A_2.

otherwise, for the set of the lowest pairwise orthogonal normalized Ξ-functions,

$$\langle \Xi_{(a,b)} \mid \Xi_{(a,b)} \rangle_M = 8M^2 \times \begin{cases} 4, & \text{if } a = 0, M \text{ and } b = 0, \\ 2, & \text{if } a = 0 \text{ and } b = \frac{M}{2}, \\ 1, & \text{if } 0 < a < M \text{ and } 0 < a + 2b < M, \\ & \text{or } a = 0 \text{ and } 0 < b < \frac{M}{2}, \\ & \text{or } a + 2b = M \text{ and } 0 < b < \frac{M}{2}, \end{cases}$$

with the higher Ξ-functions repeating the values of the lowest ones.

6.2. Discretization in the case of A_2

First we describe the grid F_M^e as in (5.10). The highest root of A_2 is $\alpha_1 + \alpha_2$, therefore

$$F_M^e \stackrel{\text{def}}{=} \left\{ \left(\frac{s_1}{M}, \frac{s_2}{M} \right) \middle| s_0, s_1, s_2 \in \mathbb{Z}^{\geqslant 0}, s_0 + s_1 + s_2 = M > 0 \right\}$$
$$\cup \left\{ \left(\frac{-s_1}{M}, \frac{s_2 + s_1}{M} \right) \middle| s_1 \geqslant 0 \right\}.$$

See figure 6 for F_M^e, $M = 3, 4$.

The coefficients c_s for A_2 are found in [4]. By (5.9)

$$\varepsilon_s \equiv \varepsilon_{\left(\frac{s_1}{M}, \frac{s_2}{M}\right)} = \begin{cases} \frac{1}{2} & \text{if } s_2 = 0 \text{ and } s_1 = M, \\ & \text{or } s_2 = M \text{ and } s_1 = -M, \\ 1, & \text{if } s_1 = 0 \text{ and } s_2 = 0, M, \\ \frac{3}{2}, & \text{if } s_2 = 0 \text{ and } 0 < s_1 < M, \\ & \text{or } s_2 = M \text{ and } -M < s_1 < 0, \\ & \text{or } s_1 + s_2 = M \text{ and } 0 < s_2 < M, \\ & \text{or } s_1 + s_2 = 0 \text{ and } 0 < s_2 < M, \\ 3, & \text{if } 0 < s_2 < M \text{ and } 0 < s_1 + s_2 < M. \end{cases} \tag{6.3}$$

The finite set $\Lambda_M = \{(a, b) \in P_e \,|0 < a + b \leqslant M, 0 \leqslant a < M\}$. For any $(a, b) \neq (a', b') \in \Lambda_M$,

$$\langle \Xi_{(a,b)} \mid \Xi_{(a',b')} \rangle_M = 0,$$

otherwise, for the set of the lowest pairwise orthogonal normalized Ξ-functions,

$$\langle \Xi_{(a,b)} \mid \Xi_{(a,b)} \rangle_M = 9M^2 \times \begin{cases} 1, & \text{if } 0 < a < M \text{ and } b = 0, M, \\ & \text{if } 0 \leqslant a + b \leqslant M \text{ and } 0 < a < M, \\ 3, & \text{if } a = 0 \text{ and } b = 0, \\ & \text{or } a = 0 \text{ and } b = M, \\ & \text{or } a = M \text{ and } b = 0, \end{cases}$$

with the higher Ξ-functions repeating the values of the lowest ones.

6.3. Discretization in the case of G_2

Since the highest root of G_2 is $2\alpha_1 + 3\alpha_2$ the grid F^e_M is

$$F^e_M \overset{\text{def}}{=} \left\{ \left(\frac{s_1}{M}, \frac{s_2}{M} \right) \middle| s_0, s_1, s_2 \in \mathbb{Z}^{\geqslant 0}, s_0 + 2s_1 + 3s_2 = M > 0 \right\} \\ \cup \left\{ \left(\frac{2s_1 + 3s_2}{M}, \frac{-s_2}{M} \right) \middle| s_2 \geqslant 0 \right\}.$$

The coefficients c_s for G_2 are found in [3]. By (5.9)

$$\varepsilon_s \equiv \varepsilon_{\left(\frac{s_1}{M}, \frac{s_2}{M}\right)} = \begin{cases} 1, & \text{if } s_1 = 0 \text{ and } s_2 = 0, \\ & \text{or } s_1 = 0 \text{ and } s_2 = \frac{M}{3}, \\ & \text{or } s_1 = M \text{ and } s_2 = -\frac{M}{3}, \\ 3, & \text{if } s_1 = 0 \text{ and } 0 < s_2 < \frac{M}{3}, \\ & \text{or } 3s_2 + s_1 = 0 \text{ and } 0 < s_1 < M, \\ & \text{or } 3s_2 + 2s_1 = M \text{ and } 0 < s_1 < M, \\ 6, & \text{if } 0 < s_1 < M \text{ and } 0 < s_1 + 3s_2 \text{ and } 2s_1 + 3s_2 < M. \end{cases} \tag{6.4}$$

The finite set $\Lambda_M = \left\{(a, b) \in P_e \,|\, 0 < 3a + b, 3a + 2b \leqslant M, 0 \leqslant a \leqslant \frac{M}{3}\right\}$. Then for any $(a, b) \neq (a', b') \in \Lambda_M$,

$$\langle E_{(a,b)} \mid E_{(a',b')} \rangle_M = 0,$$

otherwise, for the set of the lowest pairwise orthogonal normalized E-functions,

$$\langle E_{(a,b)} \mid E_{(a,b)} \rangle_M = 6M^2 \times \begin{cases} 6, & \text{if } a = 0 \text{ and } b = 0, \\ 3, & \text{if } a = \frac{M}{3} \text{ and } b = 0, \\ 2, & \text{if } a = 0 \text{ and } b = \frac{M}{2}, \\ 1, & \text{or } b = 0 \text{ and } 0 < 3a < M, \\ & \text{if } a = 0 \text{ and } 0 < 2b < M, \\ & \text{or } 0 < 3a + b \text{ and } 3a + 2b < M, \end{cases}$$

with the higher E-functions repeating the values of the lowest ones.

7. Decomposition of products of *E*-functions

For any two points $\lambda, \lambda' \in P_e$ define the product of corresponding orbits $W_e(\lambda) \otimes W_e(\lambda')$ as a set of all points in $\mathbb{R}^n$ of the form $\mu + \mu', \mu \in W_e(\lambda), \mu' \in W_e(\lambda')$. Since the set of the points $\mu + \mu'$ $\mu \in W_e(\lambda), \mu' \in W_e(\lambda')$ is invariant under the action of corresponding even Weyl group, for any $\gamma \in W_e(\lambda) \otimes W_e(\lambda')$ we obtain that

$$W_e(\gamma) \subset W_e(\lambda) \otimes W_e(\lambda'). \tag{7.1}$$

Therefore any product of two orbits can be seen as a union of finite number of orbits of W_e.

Let both λ and λ' be in P_e and

$$W_e(\lambda) \otimes W_e(\lambda') = \bigcup_{\gamma \in I} W_e(\gamma), \tag{7.2}$$

where I is a finite subset of P_e. Then for the product of corresponding E-functions we have

$$E_\lambda(x) E_{\lambda'}(x) = \sum_{\gamma \in I} E_\gamma(x). \tag{7.3}$$

Indeed, by (7.2)

$$E_\lambda(x)E_{\lambda'}(x) = \sum_{\mu\in W_e(\lambda)} \mathrm{e}^{2\pi\mathrm{i}\langle\mu,x\rangle} \sum_{\mu'\in W_e(\lambda')} \mathrm{e}^{2\pi\mathrm{i}\langle\mu',x\rangle} = \sum_{\gamma\in I}\ \sum_{o(\gamma)\in W_e(\gamma)} \mathrm{e}^{2\pi\mathrm{i}\langle o(\gamma),x\rangle} = \sum_{\gamma\in I} E_\gamma(x).$$

For the E-functions of A_1 the product decomposition was shown in (3.9). However, in higher dimensions of Euclidean spaces the problem of finding terms of the sum and their multiplicities in (7.3) is a not simple task. Further in this section we deal with the decomposition of the product of E-functions for the three simple Lie groups of rank 2.

7.1. Decomposition of products of E-functions for C_2

For any $(a, b), (c, d) \in P_e$ the product in (7.3) can be written as

$$\begin{aligned}\Xi_{(a,b)}\Xi_{(c,d)} &= \Xi_{(a+c,b+d)} + \Xi_{(a-c,b-d)} + \Xi_{(a+2d+c,b-c-d)} + \Xi_{(a-2d-c,b+d+c)} \\ &= \sum_{\mu\in W_e(c,d)} \Xi_{(a,b)+\mu}.\end{aligned} \tag{7.4}$$

Here we again we use the normalization (4.5). Moreover, we can obtain analogous product decomposition rules for C-function of the Lie group C_2. For that we first introduce analogous renormalization of orbit functions. Namely,

$$\Omega_\lambda(x) = \frac{|W|}{|W(\lambda)|} C_\lambda(x). \tag{7.5}$$

Then from (3.4) we obtain

$$\begin{aligned}&\Omega_{(a,0)} = \Xi_{(a,0)}, \qquad \Omega_{(0,b)} = \Xi_{(0,b)}, \quad a, b \neq 0,\\ &\Omega_{(a,b)} = \Xi_{(a,b)} + \Xi_{(-a,a+b)}.\end{aligned} \tag{7.6}$$

Combining both (7.6) and (7.4) we obtain the formulae for the product of C-functions. If $(a, b) \in P^{++}$ and $(c, d) \in P^{+}\backslash(0, 0)$

$$\Omega_{(a,b)}\Omega_{(c,d)} = \sum_{\mu\in W(c,d)} \frac{|W|}{|W((a,b)+\mu)|}\Omega_{(a,b)+\mu}. \tag{7.7}$$

The product of two C-functions determined by points (a, b) and (c, d) is decomposed into the sum of C-function labelled by weights from $(a, b) + W(c, d)$. In particular, if both (c, d) and all the points of $(a, b) + W(c, d)$ are in P^{++}, we obtain

$$\begin{aligned}\Omega_{(a,b)}\Omega_{(c,d)} = {}&\Omega_{(a+c,b+d)} + \Omega_{(a-c,b-d)} + \Omega_{(-a+c,a+b+d)} + \Omega_{(-a-c,a+b-d)}\\ &+ \Omega_{(a+c+2d,b-d)} + \Omega_{(a+c-2d,b+c+d)} + \Omega_{(a+c+2d,b-c-d)} + \Omega_{(a-c-2d,b+d)}.\end{aligned}$$

The remaining cases are

$$\begin{aligned}\Omega_{(a,0)}\Omega_{(c,0)} &= \Omega_{(a+c,0)} + \Omega_{(a-c,0)} + \frac{|W|}{|W(a-c,c)|}\Omega_{(a-c,c)},\\ \Omega_{(0,b)}\Omega_{(0,d)} &= \Omega_{(0,b+d)} + \Omega_{(0,b-d)} + \frac{|W|}{|W(2d,b-d)|}\Omega_{(2d,b-d)},\\ \Omega_{(a,0)}\Omega_{(0,d)} &= \frac{|W|}{|W(a,d)|}\Omega_{(a,d)} + \frac{|W|}{|W(a,-d)|}\Omega_{(a,-d)}.\end{aligned}$$

We can also use these formulae to generalize formulae for tensor product of W-orbits of C_2 from section 4.2 in [6].

7.2. Decomposition of products of E-functions for A_2

Analogously to the case of C_2 group products of the E-functions of A_2 decompose into sums of E-functions. For any $(a, b), (c, d) \in P_e$ we obtain

$$\Xi_{(a,b)}\Xi_{(c,d)} = \Xi_{(a+c,b+d)} + \Xi_{(a+d,b-c-d)} + \Xi_{(a-c-d,b+c)} = \sum_{\mu\in W_e(c,d)} \Xi_{(a,b)+\mu}. \tag{7.8}$$

Relation (3.4) between E- and C-functions for A_2 is the following:

$$\begin{aligned} &\Omega_{(a,0)} = \Xi_{(a,0)}, \qquad \Omega_{(0,b)} = \Xi_{(0,b)} \\ &\Omega_{(a,b)} = \Xi_{(a,b)} + \Xi_{(-a,a+b)}. \end{aligned} \tag{7.9}$$

Here we used the normalization (7.5) for C-functions. Combining the last two formulae we obtain for any $(a, b) \in P^{++}$ and any $(c, d) \in P^{+}\backslash(0, 0)$

$$\Omega_{(a,b)}\Omega_{(c,d)} = \sum_{\mu\in W(c,d)} \frac{|W|}{|W((a,b)+\mu)|}\Omega_{(a,b)+\mu}. \tag{7.10}$$

The remaining cases:

$$\begin{aligned} \Omega_{(a,0)}\Omega_{(c,0)} &= \Omega_{(a+c,0)} + \frac{|W|}{|W(a,-c)|}\Omega_{(a,-c)}, \\ \Omega_{(0,b)}\Omega_{(0,d)} &= \Omega_{(0,b+d)} + \frac{|W|}{|W(-d,b)|}\Omega_{(-d,b)}, \\ \Omega_{(a,0)}\Omega_{(0,d)} &= \frac{|W|}{|W(a,d)|}\Omega_{(a,d)} + \Omega_{(0,-a+d)}. \end{aligned}$$

Also we have obtained the formulae generalizing formulae for tensor product of W-orbits of A_2 from section 4.2 in [6].

7.3. Decomposition of products of E-functions for G_2

The products of the E-functions decompose into sums of E-functions. Namely, if $(a, b), (c, d) \in P_e$

$$\begin{aligned} \Xi_{(a,b)}\Xi_{(c,d)} = \sum_{\mu\in W_e(c,d)} \Xi_{(a,b)+\mu(c,d)} &= \Xi_{(a+c,b+d)} + \Xi_{(a-c,b-d)} + \Xi_{(a+2d+c,b-3c-d)} \\ &\quad + \Xi_{(a-2d-c,b+3c+d)} + \Xi_{(a-c-d,b+3c+2d)} + \Xi_{(a+c+d,b-3c-2d)}. \end{aligned} \tag{7.11}$$

In the case of G_2 (3.4) gives

$$\begin{aligned} &\Omega_{(a,0)} = \Xi_{(a,0)}, \qquad \Omega_{(0,b)} = \Xi_{(0,b)} \\ &\Omega_{(a,b)} = \Xi_{(a,b)} + \Xi_{(-a,3a+b)}. \end{aligned} \tag{7.12}$$

As a sequence from (7.11) and (7.12) we obtain the decomposition for the product of Ω for $(a, b) \in P^{++}$ and $(c, d) \in P^{+}\backslash(0, 0)$:

$$\Omega_{(a,b)}\Omega_{(c,d)} = \sum_{\mu\in W(c,d)} \frac{|W|}{|W((a,b)+\mu)|}\Omega_{(a,b)+\mu}. \tag{7.13}$$

8. Central splitting for *E*-transforms

The idea of the central splitting of a function $f(x)$ on F or F^e, of a compact semisimple Lie group G, is the decomposition of $f(x)$ into the sum of several component functions, as many as is the order s of the centre Z of G. Motivation for considering such splitting is in the property of the component functions [2]: their E-transforms employ mutually exclusive subsets of E-functions of G. The functions E_λ and $E_{\lambda'}$ belong to the same subset precisely if λ and λ' belong to the same congruence class.

Let $\chi_1, \ldots, \chi_s$ be the irreducible characters of Z. Also any $\lambda \in P_e$ determines an irreducible character of Z:

$$\chi_\lambda : \check{z} \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle\lambda,\check{z}\rangle} \qquad \check{z} \in Z. \tag{8.1}$$

Then $\chi_\lambda = \chi_j$ for some $1 \leqslant j \leqslant s$. Then j is called the congruence class of λ. It is constant on the W_e-orbit of λ and therefore

$$E_\lambda(x + \check{z}) = \chi_j(\check{z}) E_\lambda(x) \qquad \check{z} \in Z. \tag{8.2}$$

Thus any f which is linear combination of E-functions can be written as a sum of s functions $f(x) = f_1 + \cdots + f_s$, where

$$f_i(x) = \frac{1}{s} \sum_{\check{z}\in Z} \overline{\chi_i(\check{z})} f(x + \check{z}) \qquad 1 \leqslant i \leqslant s. \tag{8.3}$$

There are two rank-two compact simple Lie groups with non-trivial centre of orders 2 and 3, namely C_2 and A_2, respectively. The centre of G_2 is trivial.

8.1. Central splitting for C_2

As we have already seen in (4.3) the fundamental region of C_2 is a square with vertices 0, $\frac{\check{\omega}_1}{2}$, $\check{\omega}_2$ and $\check{\omega}_2 - \frac{\check{\omega}_1}{2}$. The centre Z has two elements $\{0, \check{\omega}_2\}$. According to [2] any function f on F^e we may decompose it into $f(x) = f_0(x) + f_1(x)$, where

$$f_0(x) = \tfrac{1}{2}\{f(x) + f(x + \check{\omega}_2)\} \qquad \text{and} \qquad f_1(x) = \tfrac{1}{2}\{f(x) - f(x + \check{\omega}_2)\}. \tag{8.4}$$

However for any $x = a\check{\omega}_1 + b\check{\omega}_2 \in F^e$ the point $x + \check{\omega}_2$ is outside of F^e. By suitable transformation we bring it back to the fundamental region:

$$r_{\alpha_2,1} r_{\alpha_1} r_{\alpha_2,1} r_{\alpha_1}(a, b + 1) = (-a, 1 - b).$$

Therefore, the component function can be written for x in $\check{\omega}$-basis as

$$f_0(a, b) = \tfrac{1}{2}\{f(a, b) + f(-a, 1 - b)\} \qquad f_1(a, b) = \tfrac{1}{2}\{f(a, b) - f(-a, 1 - b)\}.$$

The main property of both f_0 and f_1 is that each of them decomposes into a linear combination of E-functions from one congruence class only 0 for f_0 and 1 for f_1.

8.2. Central splitting for A_2

In the case of A_2, the fundamental region is a rhombus with vertices $0, \omega_1, \omega_2$ and $\omega_2 - \omega_1$ as in (4.8). The centre Z has three elements $\{0, \omega_1, \omega_2\}$ and any function on F^e is decomposed into the sum of three function $f(x) = f_0(x) + f_1(x) + f_2(x)$ where

$$\begin{aligned}
f_0(x) &= \tfrac{1}{3}\{f(x) + f(x + \omega_1) + f(x + \omega_2)\}, \\
f_1(x) &= \tfrac{1}{3}\{f(x) + \mathrm{e}^{-2\pi\mathrm{i}/3} f(x + \omega_1) + \mathrm{e}^{-2\pi\mathrm{i}/3} f(x + \omega_2)\}, \\
f_2(x) &= \tfrac{1}{3}\{f(x) + \mathrm{e}^{2\pi\mathrm{i}/3} f(x + \omega_1) + \mathrm{e}^{2\pi\mathrm{i}/3} f(x + \omega_2)\}.
\end{aligned}$$

Again for $x = a\omega_1 + b\omega_2 \in F^e$ both $x + \omega_1$ and $x + \omega_2$ are not necessarily in F^e. We have $x+\omega_1 = (a+1, b)$ and $x+\omega_2 = (a, b+1)$. Then there are two cases for points $(a, b+1)$ and $(a+1, b)$ to be brought to F^e by suitable transformations from the affine Weyl group of A_2:

$$\begin{aligned} r_{\alpha_2,1} r_{\alpha_1}(a, b+1) &= (b, -a-b+1) \\ r_{\alpha_1,1} r_{\alpha_2}(a+1, b) &= (1-a-b, a) \end{aligned} \qquad \text{for} \quad a \geqslant 0$$

and

$$\begin{aligned} r_{\alpha_1} r_{\alpha_2,1}(a, b+1) &= (b-1, a+1) \\ r_{\alpha_2,1} r_{\alpha_1}(a+1, b) &= (b-1, -a-b+1) \end{aligned} \qquad \text{for} \quad a < 0.$$

Finally, one obtains, for $x = (a, b) \in F^e$, $a \geqslant 0$,

$$\begin{aligned} f_0(a,b) &= \tfrac{1}{3}\{f(a,b) + f(b-1, -a-b+1) + f(b-1, a+1)\}, \\ f_1(a,b) &= \tfrac{1}{3}\{f(a,b) + \mathrm{e}^{-2\pi \mathrm{i}/3} f(b-1, -a-b+1) + \mathrm{e}^{-2\pi \mathrm{i}/3} f(b-1, a+1)\}, \\ f_2(a,b) &= \tfrac{1}{3}\{f(a,b) + \mathrm{e}^{2\pi \mathrm{i}/3} f(b-1, -a-b+1) + \mathrm{e}^{2\pi \mathrm{i}/3} f(b-1, a+1)\}, \end{aligned}$$

or if $x = (a, b) \in F^e$, $a < 0$

$$\begin{aligned} f_0(a,b) &= \tfrac{1}{3}\{f(a,b) + f(1-a-b, a) + f(b, -a-b+1)\}, \\ f_1(a,b) &= \tfrac{1}{3}\{f(a,b) + \mathrm{e}^{-2\pi \mathrm{i}/3} f(1-a-b, a) + \mathrm{e}^{-2\pi \mathrm{i}/3} f(b, -a-b+1)\}, \\ f_2(a,b) &= \tfrac{1}{3}\{f(a,b) + \mathrm{e}^{2\pi \mathrm{i}/3} f(1-a-b, a) + \mathrm{e}^{2\pi \mathrm{i}/3} f(b, -a-b+1)\}. \end{aligned}$$

Again each f_i, $i = 0, 1, 2$, in this sum may be decomposed as a sum of E-functions from the congruence class i.

9. Concluding remarks

(1) Similarly as the C- and S-functions, the E-functions can be viewed as a family of orthogonal polynomials, related to a particular semisimple Lie group and to a particular W_e-orbit, in as many variables as is the rank of the group. Such variables, as we use them here, are constrained to the n-dimensional torus of the appropriate Lie group. The polynomials have many properties of traditional special functions. Easy discretization of the polynomials is an unusual feature, particularly in a multidimensional setup.

(2) The E-functions of $SU(2)$ are common exponential function in one variable

$$E_m(x) = \mathrm{e}^{\mathrm{i}mx} = \tfrac{1}{2}(C_m(x) + S_m(x)).$$

Roughly speaking, E-function are related to C- and S-functions as the exponential function is related to cosine and sine functions. The special role of imaginary unit does not seem to generalize.

(3) Besides three introduced transforms in the case when G is semisimple there are other derived transforms which may be considered. Suppose $G = G_1 \times G_2$, where G_1 and G_2 are simple (semisimple) Lie groups. Let F_1 be either the fundamental region of G_1 or fundamental region of its even subgroup and F_2 respectively for G_2. Then any function $f(x_1, x_2)$ on $F_1 \times F_2$ can be expanded using any of C- S-, E-functions on F_1 and any of these three types on F_2. Thus one may have EC- or SE-transforms rather then EE-transforms we studied in the main body of the paper.

(4) The E-functions are complex valued, in general. A function $E_\lambda(x)$ is real precisely if the orbit $W_e(\lambda)$ contains both weights $\pm\lambda$. More about when that happens see in [2].

(5) A choice of W_e^{aff}-fundamental domain F^e is not unique. It is made out of two adjacent copies of the fundamental domain F of W. One can flip F in any of its $(n-1)$-dimensional faces in order to get F^e. Obviously, for any choice of F^e one can introduce both continuous and discrete transforms: general theory allows one to set up P_e, F_M^e, etc. It is conceivable that practical considerations may dictate preferred choice.

(6) As we already mentioned in the introduction there are two ways to define even Weyl group in the case when original Lie group is a product of two simple Lie groups $G = G_1 \times G_2$. First possibility is when E-transform of G is taken up to be the simultaneous E-transform of G_1 and the E-transform of G_2. The E-functions of G are products of E-functions of G_1 and the E-functions of G_2. In this case, we take $W_e(G)$ to be $W_e(G_1) \times W_e(G_2)$. The second possibility arises from the fact that

$$W_e(G_1 \times G_2) \neq W_e(G_1) \times W_e(G_2). \tag{9.1}$$

Consider for example $G = SU(2) \times SU(2)$. Its Weyl group is of order 4, its elements being $1, r_1, r_2, r_1r_2$. Hence $W_e(SU(2) \times SU(2))$ has two elements, namely 1 and r_1r_2. In three dimensions the possibilities this option allow are more curious, interesting and involved. We are going to pursue them elsewhere.

(7) The most important qualitative argument in favour of efficiency of discrete and continuous expansions of functions given on F into series of either C- or S- or E-functions is that they involve discrete groups larger than the translation group of traditional Fourier expansions. Indeed, it is the affine Weyl group acting in $\mathbb{R}^n$, which contains the translations as its subgroup. More specifically, the fundamental region of the translation group is the proximity cell (Voronoi domain) V of the root lattice of G, while the fundamental region F for the affine Weyl group is much smaller, $|V| = |F||W|$. Thus, the larger is the order $|W|$ of the Weyl group, the more efficient are our expansions (fewer 'harmonics' needed). The Voronoi domains of root lattices for all simple G are described in [14].

Independently interesting would be to study multidimensional Fourier expansions in general, that is expansions based on translational symmetry, as opposed the reflection symmetry of the affine Weyl group we use. In that case Voronoi domains would play the role of F here, because they are the tiles filling the space by translations. Suppose that one wants to insists on expansions based on translation symmetries like $x \mapsto x + 2\pi$ in one dimension. Then the corresponding symmetry group is the translation subgroup of the affine Weyl group W^{aff}. The expansions then refer to functions given on the fundamental region of W^{aff} which is proximity cell (Voronoi domain) of the root lattice of G. Translations then tile the entire n-dimensional space by copies of the proximity cell. A description of the cells for all simple Lie groups is found in [6].

(8) Finally, let us point out several questions naturally arising from this work and its possible extensions. There are two E-transforms on square lattices of $\mathbb{R}^2$ related to the groups $SU(2) \times SU(2)$ and $O(5)$. How do they compare? Similarly there is E-transforms on triangular lattices of $G(2)$ and C-transform on the same lattice of $SU(3)$. When to use one and when the other? Such dilemmas grow rapidly with the dimension of the transform. Thus in 3D there are four E-transforms on cubic lattices.

To know more about computing efficiency of the transforms would be very useful.

Restriction of the Lie group G to, say, its maximal reductive subgroup G' implies reduction of the E-functions of G to the sum of E-functions of G'. Calculate such branching rules.

There are finitely few discrete points in F (for each semisimple G) where all C-functions take integer values. Are there points with this property also for E-functions? A trivially affirmative answer is given by $E_\lambda(0)$ for all λ and all G.

Symmetrization and antisymmetrization of tensor powers of W_e orbits result in the sum of several orbits. In terms of E-functions such an uncommon multiplication would yield a sum of E-functions. Any use for it?

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